An algorithm for computing partial derivatives of the value function by simulation

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Abstract

The problems involving incentive compatibility constrains in the form of participation constrains have received wide attention in the literature due to the recent advances in the dynamic optimization techniques. Often the optimality conditions for this class of problems involve partial derivatives of the optimal value function with respect to some of the endogenous state variables. In this note we suggest an algorithm for computing these partial derivatives by simulation. The attractive features of the algorithm include its rather wide scope of applicability and simplicity of implementation. Furthermore, the suggested method does not suffer from the curse of dimensionality and therefore it is particularly convenient for the models involving many state variables. (JEL C63, E10)

Keywords: dynamic participation constrains, numerical algorithm, simulation.

1 Introduction

The purpose of this note is to propose a simple algorithm for computing partial derivatives of the optimal value function. The problems involving incentive compatibility constrains in the form of participation constrains have received wide attention in the literature due to the recent advances in the dynamic optimization techniques. Often the optimality conditions for this class of problems involve partial derivatives with respect to some of the endogenous state variables of the optimal value function corresponding to the dynamic programming formulation of an outside option. Although many numerical methods can provide an approximation for the value function, there is no reason to believe that a derivative of this approximation will be close in any sense to the actual value of the derivative. In this note we suggest an algorithm for computing these partial derivatives by simulation.

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This issue has been previously considered by Marcet and Marimon (1992) who solved numerically a growth model with capital accumulation under one-sided lack of commitment. To circumvent the problem of finding the values of the derivatives Marcet and Marimon (1992) proposed a rather convenient method based on the ideas of Benveniste and Scheinkman (1979). Unfortunately their methods has limited applicability since it depends on the availability of the analytical solution for the derivatives as conditional expectations of the known functions of the model solution. In this note we propose a simple algorithm to fill this gap.

To demonstrate the main idea of the algorithm consider a typical example of a partnership with limited commitment\(^1\). Suppose that the enforcement technology available is such that if the agent deviates from the optimal plan he has no choice but to switch to autarky and stay there forever. In this case the reservation value for the agent would be the value function of the autarkic solution \(V(x_t, s_t)\) evaluated at the values of the endogenous state variables \(x_t\), and exogenous shocks \(s_t\) at the moment of deviation. The optimality conditions for the planners problem will involve the values of the partial derivatives of the value function \(V(x_t, s_t)\) with respect to some of the endogenous state variables. As is common in the endogenous incomplete markets literature, the intertemporal conditions (Euler equations) explicitly take into account the effect the changes in the values of state variables will have on the agent’s incentives to deviate from the optimal plan. In the discussion that follows we will assume that these partial derivatives of the optimal value function actually exist\(^2\).

In order to be able to use finite differences to approximate the values of the partial derivatives at a given point one would need to know the values of the optimal value function at a certain set of points. The numerical procedure we sketch below allows one to obtain approximations of these values with arbitrary precision. Moreover, achieving this accuracy is feasible for all points in the state-space which have economic relevance.

The initial step of the algorithm involves obtaining numerical solution to the autarkic problem using a procedure which satisfies three criteria. First, it approximates some unknown function with flexible functional forms of finite elements. Second, it can deliver an accurate solution as the number of the finite elements in the function goes to infinity. Finally, the resulting numerical solution must be such that it can be formulated as a set of policy functions approximated with flexible functional forms. In order to execute this step we will rely on a version of the parameterized expectation approach (PEA)^3. Even though several procedures suit well for our purpose, the choice of PEA can be justified due to it’s following inherent features. First, PEA is computationally efficient when there is a large number of state variables and stochastic shocks in the conditional expectations\(^4\). Furthermore, it does

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\(^1\)Examples of such models include extensions of the economies in Kocherlakota (1996) and Alvarez and Jermann (2000) to a context with endogenous production as in Kehoe and Perri (2004, 2002) or Abraham and Carceles-Poveda (2006)

\(^2\)In practical applications one would most often need to establish a stronger property than merely existence of the partials. The conditions which ensure differentiability of the value function or at least some of its restrictions are problem specific. Some of these conditions are considered by Benveniste and Scheinkman (1979) for the cases where the Bellman equation is satisfied and by Koeppl (2006) where it is not.

\(^3\)The method was originally introduced by Marcet (1989). The modifications that followed differ along several dimensions. These include the manner the conditional expectations are parameterized, and the way the search for a fixed point is carried out. Which particular version would prove to be more suitable in terms of accuracy and computational efficiency for each class of problems is a question deserving further attention.

\(^4\)This can be partially attributed to the fact that PEA incorporates both endogenous oversampling and Monte-Carlo integration.
not impose a discrete grid on the endogenous state variables or the stochastic shocks. Second, Christiano and Fisher (2000, p. 1181) "describe PEA...as at least as accurate as all the other algorithms considered". Third, Monte Carlo integration in both evaluation of expected discounted returns and the numerical solution to the model with PEA are convenient for parallelizing.\footnote{See, for instance, Creel (2005) for a discussion of parallelizing of Monte Carlo problems.} Finally, the algorithm can be utilized to compute the transition towards the stationary distribution. Although the basic version of PEA discussed in den Haan and Marcet (1990) only suits to approximate the solution at the stationary distribution, the algorithm can be modified to include exogenous oversampling and solve for the transition (see Christiano and Fisher (2000) and Marcet and Marimon (1992) for further details).

The next step involves using Monte-Carlo integration in order to evaluate the conditional expectation of the discounted sum of future instantaneous utilities. This requires simulating a large number of realizations of the shocks using the law of motion for the exogenous state variables and their initial values. The initial values in question are simply the values of the state variables at which the value function is being approximated. The simulated series for the shocks along with the obtained approximated policy functions can subsequently be used to generate corresponding series of the endogenous variables consistent with the optimality and feasibility conditions. The obtained series will serve for calculating the discounted sums of the instantaneous returns, each of which would correspond to a particular realization of the stochastic process. An approximation of the value function at a given point is the outcome of averaging over the obtained discounted sums of the instantaneous returns.

The final step involves applying the method of finite differences to approximate the values of the partial derivatives at the point in question. Depending on the underlying assumptions about differentiability of the optimal value function one can choose to rely on several finite difference formulas to compute the derivatives.

The attractive features of the algorithm include its rather wide scope of applicability and simplicity of implementation. It can be used to study the questions of risk sharing under imperfect enforcement of contracts, as well as partnerships with limited commitment when several state variables appear in the model corresponding to the outside option. Such models may include habit formation preferences, several types of capital, or reputational co-state variables. Furthermore, the method may still be applicable even though the default model may fail to fall into a standard recursive framework. Furthermore, the suggested method is computationally inexpensive, it does not suffer from the curse of dimensionality and therefore it is particularly convenient for the models involving many state variables.

The rest of the paper is organized as follows. In Section 2 we sketch the idea behind the algorithm. In Section 3 we suggest some examples where the proposed algorithm will prove to be useful. In Section 4 we offer a worked out example of implementation of the algorithm and relate the algorithm with the available alternatives. Section 5 concludes.

## 2 The Algorithm

Typically, in the models with participation constraints the reservation value is the value function of the outside alternative evaluated at the current values of the endogenous state variables, \( x \), and exogenous shocks, \( \sigma \). Suppose, that the value of the optimal value function
at a point \((\bar{x}, \bar{s})\) is an outcome of a standard optimization problem for the outside alternative, which can be written as follows:

\[
V(\bar{x}, \bar{s}) = \max_{\{a_t\}} E_0 \sum_{t=0}^{\infty} \beta^t r(x_t, a_t, s_t)
\]

subject to \(x_{t+1} = l(x_t, a_t, s_t), \quad a_t \in A(x_t, s_t)\),

\[
x_0 = \bar{x}, \quad s_0 = \bar{s},
\]

where \(r\) is an instantaneous utility function, \(\beta \in (0, 1)\) the discount factor, \(\{s_t\}\) an exogenous Markov stochastic process, \(x_t\) a vector of endogenous state variables, \(a_t\) a vector of control variables, \(A\) a feasibility correspondence and \(l\) the law of motion for the endogenous state variables. The functional equation to this problem can be derived using the standard dynamic programming techniques. It yields a time invariant policy function \(f\) such that optimal allocations satisfy \(a_t = f(x_t, s_t)\).

The purpose of the suggested algorithm is to find an pointwise approximation to the partial derivatives \(\frac{\partial}{\partial x_i} V(\bar{x}, \bar{s})\) of the value function with respect to its \(i\)-th argument. The algorithm takes the following three steps:

**Step I. (Numerical Solution)** Solve the model in (1) with a spectral method and formulate the solution in terms of approximated policy functions

\[
a_t = \hat{f}(\omega; x_t, s_t).
\]

**Step II. (Monte Carlo Integration).** Simulate \(N\) sequences of the realizations of the stochastic process \(\{s^n\}_{t=1}^T\) of size \(T\) with a starting value \(s_0^n = \bar{s}\), for all \(n = 1, \ldots, N\). For each sequence \(\{s^n\}_{t=0}^T\) simulate the series of the endogenous variables \(\{x^n_t, a^n_t\}_{t=0}^T\) using approximated policy functions (3), the equations for motion for the state variables (2), and the initial values \(x^n_0 = \bar{x}\). Using the simulated series calculate the discounted sums of the instantaneous returns and average over \(N\).

\[
V(\bar{x}, \bar{s}) \simeq \frac{1}{N} \sum_{n=1}^N \sum_{t=0}^T \beta^t r(x^n_t, a^n_t, s^n_t).
\]

**Step III. (Numerical Differentiation)** Repeat Step II to obtain approximations of the value function at two points, for instance \(V(\bar{x} + \epsilon \mathbf{e}_i, \bar{s})\) and \(V(\bar{x} - \epsilon \mathbf{e}_i, \bar{s})\), where \(\mathbf{e}_i\) denotes a conformable vector of zeros with one on its \(i\)-th coordinate, and \(\epsilon\) is a small positive number. Calculate the value of the partial derivative using, for example, Stirling’s finite difference formula:

\[
\frac{\partial}{\partial x_i} V(\bar{x}, \bar{s}) \simeq \frac{V(\bar{x} + \epsilon \mathbf{e}_i, \bar{s}) - V(\bar{x} - \epsilon \mathbf{e}_i, \bar{s})}{2\epsilon}.
\]

The optimal choice of the method for calculating the derivatives in Step III it problem specific and its accuracy depends on the smoothness of the value function. The approaches
available include a variety of difference formulas, Richardson Extrapolation, or curve fitting with cubic splines. These are described at length in the standard numerical methods texts such as Judd (1998), Mathews and Fink (2004), or Press, Teukolsky, Vetterling, and Flannery (1992).

A brief note should be made at this point on the accuracy of the algorithm. In principle, arbitrary accuracy of the approximation can be achieved, by simultaneously increasing the dimension of the approximating family of functions in Step I, increasing the size of Monte Carlo iterations in Steps I and II, and decreasing the denominator $\epsilon$ in Step III. However, in practical applications there are several sources of the approximation errors. First, in order to obtain the values of optimal value function at a point one relies on the approximations of the policy functions implied by the numerical solution to the model. Second, since we consider stochastic models, there is an additional error stemming from the evaluation of the integral in computation of expected discounted returns. Finally, numerical differentiation introduces two more sources of error: the truncation error and the roundoff error. The truncation error comes from omitting higher order terms in the Taylor series expansion. The roundoff error is associated with storing real numbers in computer’s floating-point format. We will discuss some practical accuracy issues in the context of an example in section 4.

3 Some Examples of Applicability of the Algorithm

In this section we provide a number of examples where the proposed algorithm will proof to be useful. A common feature of all these examples, is that solving them boils down to designing an optimal social contract which takes into account not only technological but also incentive and legal constraints.

3.1 Risk-Sharing with Endogenous Market Incompleteness and Habit Formation Preferences

Here we present a model of international risk sharing which distinguishes itself from the celebrated model of Backus, Kehoe, and Kydland (1992) in two respects. First, following Kehoe and Perri (2002) we introduce a friction in the credit markets. We assume that the international loans are feasible only to the extent to which they can be enforced by the threat of exclusion from participating in any other international risk sharing arrangement. Second, we incorporate habit formation preference in the model. The latter can be motivated by the evidence presented by Fuhrer and Klein (2006, p. 722) who suggest that "habit formation characterizes consumption behavior amongst most of G-7 countries".

Consider the planner’s problem of maximizing a weighted sum of utilities subject to individual participation constraints and feasibility constraints. In particular, the problem of interest is to choose allocations $\{c_{it}, t_{it}\}$ for $i = 1, \ldots, I$ to solve

$$\max_{\{c_{it}, t_{it}\}} \sum_{i=1}^{I} \lambda_i \sum_{t=0}^{\infty} \beta^t u(c_{it}, h_{it})$$
subject to

\[ \sum_{i=1}^{I} c_{it} + \sum_{i=1}^{I} i_{it} = \sum_{i=1}^{I} f(k_{it}, \theta_{it}) , \]  

subject to

\( \sum_{i=1}^{I} c_{it} + \sum_{i=1}^{I} i_{it} = \sum_{i=1}^{I} f(k_{it}, \theta_{it}) , \)  

\( k_{it+1} = (1 - \delta)k_{it} + i_{it} , \)  

\( h_{it+1} = h_{it} + \lambda(c_{it} - h_{it}) , \)  

\[ E_{t} \sum_{j=0}^{\infty} \beta^{j} u(c_{it+j}, h_{it+j}) \geq V^{a}_{i}(k_{it}, h_{it}, \theta_{it}) , \]  

where \( c_{it}, i_{it} \geq 0, \theta_{t} \) follows a first order vector autoregressive process, and the initial values for the state variables \( k_{0}, h_{0}, \theta_{0} \), and the initial non-negative weights are given. Here \( V^{a}_{i}(k_{it}, h_{it}, \theta_{it}) \) denotes the optimal value function corresponding to the autarkic environment

\[ \max_{\{c_{it}, i_{it}\}} E_{0} \sum_{t=0}^{\infty} \beta^{t} u(c_{it}, h_{it}) \]  

subject to

\[ c_{it} + i_{it} = f(k_{it}, \theta_{it}) , \]  

\[ k_{it+1} = f(k_{it}, \theta_{it}) - c_{it} + (1 - \delta)k_{it} , \]  

\[ h_{it+1} = h_{it} + \lambda(c_{it} - h_{it}) , \]  

with the initial values being equal to the values of the state variables \( k_{it}, h_{it}, \theta_{it} \) at the moment of deviation from the optimal plan.

In addition to the aggregate recourse constraint (4), participation constraint (7), and the equations of motion for the state variables (5)-(6) the constrained efficient allocations should satisfy the following risk sharing condition:

\[ u_{c}(i, t) + \lambda \beta E_{t} \sum_{j=0}^{\infty} \beta^{j} (1 - \lambda)^{j} \left[ \frac{\xi_{it+j+1}}{\xi_{it}} u_{h}(i, t+j+1) \right] - \frac{\mu_{it+j}^{a}}{\xi_{it}} \frac{\partial V^{a}_{i}}{\partial h_{it+j+1}} \left( i, t+j+1 \right) \]  

\[ u_{c}(s, t) + \lambda \beta E_{t} \sum_{j=0}^{\infty} \beta^{j} (1 - \lambda)^{j} \left[ \frac{\xi_{st+j+1}}{\xi_{st}} u_{h}(s, t+j+1) \right] - \frac{\mu_{st+j}^{a}}{\xi_{st}} \frac{\partial V^{a}_{i}}{\partial h_{st+j+1}} \left( s, t+j+1 \right) \]  

\[ = \xi_{st} \xi_{it} , \]  

for \( i, s = 1, ..., I \). Optimal allocation must also satisfy the intertemporal condition

\[ (\xi_{it+j+1}^{a} u_{h}(i, t+j+1) - \mu_{it+j}^{a} \frac{\partial V^{a}_{i}}{\partial h_{it+j+1}} (i, t+j+1)) + \lambda \beta E_{t} \sum_{j=0}^{\infty} \beta^{j} (1 - \lambda)^{j} \left[ \frac{\xi_{it+j+2}}{\xi_{it}} u_{h}(i, t+j+2) \right] - \frac{\mu_{it+j+2}^{a}}{\xi_{it}} \frac{\partial V^{a}_{i}}{\partial h_{it+j+2}} (i, t+2+j) \]

\[ = \beta E_{t} \left[ \frac{\xi_{it+j+1}^{a} u_{h}(i, t+j+1) - \mu_{it+j}^{a} \frac{\partial V^{a}_{i}}{\partial h_{it+j+1}} (i, t+j+1)}{\xi_{it}} \right] \]

\[ + \lambda \beta E_{t} \sum_{j=0}^{\infty} \beta^{j} (1 - \lambda)^{j} \left[ \frac{\xi_{st+j+2}}{\xi_{st}} u_{h}(s, t+j+2) \right] - \frac{\mu_{st+j+2}^{a}}{\xi_{st}} \frac{\partial V^{a}_{i}}{\partial h_{st+j+2}} (s, t+j+2) \]

\[ = \beta E_{t} \left[ \frac{\xi_{it+j+1}^{a} u_{h}(i, t+j+1) - \mu_{it+j}^{a} \frac{\partial V^{a}_{i}}{\partial h_{it+j+1}} (i, t+j+1)}{\xi_{it}} \right] , \]
along with the complementary slackness condition

\[ \mu_{it} \left[ E_t \sum_{j=0}^{\infty} \beta^j u(c_{it+j}, h_{it+j}) - V_t^a(k_{it}, h_{it}, \theta_{it}) \right] = 0, \]

and the law of motion for the reputational co-state variables

\[ M_{it+1} = M_{it} + \mu_{it}, \]

where \( \xi_{it} = \lambda_i + M_{it+1}, \mu_{it} \geq 0, \) and \( M_{i0} = 0. \) In these first order conditions we have used the abbreviations \( u_c(i, t) \) for \( \frac{\partial u(c_{it}, h_{it})}{\partial c_{it}} \), and we have used similar abbreviations for other terms. Notice, that the partial derivative of the optimal value function \( V_t^a \) enter both the intertemporal condition (Euler equation) and the risk-sharing condition.

### 3.2 Capital flows to developing countries under risk of debt repudiation.

Our next example is a dynamic principal-agent problem with one-sided lack of commitment similar in spirit to that of Marcet and Marimon (1992). In this setup, a capital-poor country, represented as a risk-averse agent, can borrow from the industrialized capital-rich countries, represented here as a risk neutral agent. While the club of the industrial countries is assumed to honor its contractual obligations, the developing country has an option to renege on its debt and suffer the consequences. In the absence of a supranational enforcement authority the punishment the borrower will incur in case of debt repudiation is the exclusion from any further intertemporal and inter-state trade with the industrialized countries. However, this will not preclude the poor country from being able to enter a risk sharing arrangement under two-sided lack of commitment with some other capital poor countries.

Constrained efficient allocations can be found as a solution to the following planner’s problem:

\[
\max_{\{c_t, \tau_t, i_t\}_{t=0}} E_0 \left[ \sum_{t=0}^{\infty} \beta^t [\lambda u(c_t) + (-\tau_t)] \right]
\]

subject to

\[ c_t - \tau_t + i_t = \theta_t f(k_t), \quad (11) \]

\[ k_{t+1} = (1 - \delta)k_t + i_t, \quad (12) \]

\[ E_t \left[ \sum_{j=0}^{\infty} \beta^j u(c_{it+j}) \right] \geq V^d(k_t, \theta_t), \quad (13) \]
with initial conditions \((k_0, \theta_0)\) and non-negativity conditions \(c_t \geq 0, i_t \geq 0\). In this specification \(c_t, i_t, \) and \(k_t\) are consumption, gross investment, and capital stock respectively; \(\tau_t\) denotes transfers from the risk-neutral agent to the risk adverse one; \(\delta \in (0, 1]\) is depreciation rate of capital; \(\beta \in (0, 1)\) is the discount factor. A random productivity shock \(\theta_t\) follows a first order stationary Markov process with bounded support. The instantaneous utility \(u(\cdot)\) of the risk-averse agent is strictly concave, twice differentiable and satisfies the Inada conditions. The production function \(f(\cdot)\) is concave and differentiable.

The reservation value which appears in left hand side of (13) corresponds to the expected felicity of the agent’s outside option: a partnership with two-sided lack of commitment. The allocation to the latter must satisfy the following planner’s problem, with \(\sum_{i=1}^{I} \lambda_i = 1\):

\[
\max_{\{c_{it}, i_{it}\}} E_0 \left\{ \sum_{i=1}^{I} \lambda_i \sum_{t=0}^{\infty} \beta^t u(c_{it}) \right\} \tag{14}
\]

subject to

\[
\sum_{i=1}^{I} c_{it} + \sum_{i=1}^{I} i_{it} = \sum_{i=1}^{I} \theta_{it} f(k_{it}),
\]

\[
k_{it+1} = (1 - \delta)k_{it} + i_{it},
\]

\[
E_t \sum_{j=0}^{\infty} \beta^j u(c_{it+j}) \geq V_i^a(k_{it}, \theta_t). \tag{15}
\]

with \(c_{it} \geq 0, i_{it} \geq 0\). The initial values for \(k_{i0}, \theta_{i0}\) are given by the corresponding values of the state variables in problem (10) at the moment of deviation. In the problem (14) above, the reservation value \(V_i^a(k_{it}, \theta_t)\) is given by

\[
V_i^a(k_{it}, \theta_t) = \max_{\{c_{is}, i_{is}\}} E \left[ \sum_{s=t}^{\infty} \beta^{s-t} u(c_{is}) \right] \tag{16}
\]

subject to

\[
k_{is+1} = \theta_{is} f(k_{is}) + (1 - \delta)k_{is} - c_{is},
\]

with \(c_{it} \in [0, \theta_{is} f(k_{is})]\).

While the autarkic problem in (16) is the classical Brock and Mirman (1972) economy the solutions to the problem (14) will take the form of the following policy function:\(^6\)

\[
\Psi(k, \mu, \theta) = \arg \min_{\gamma \geq 0} \max_{c,i} \left\{ \sum_{j=1}^{I} (\lambda_j + \mu_j + \gamma_j) u(c_j) + \beta \mathbb{E} [W(k', \mu', \theta')] \mid \theta \right\} \tag{17}
\]

\(^6\)Henceforth we adopt the dynamic programming convention by which primes denote the forwarded values of variables.
subject to
\[ \mu'_j = \mu_j + \gamma_j, \]
\[ \sum_{j=1}^{I} c_j + \sum_{j=1}^{I} i_j = \sum_{j=1}^{I} \theta_j f(k_j), \]
\[ k'_j = (1 - \delta)k_j + i_j, \]
in addition to the incentive compatibility constraint (15) and non-negativity constraints on the multipliers \( \{\mu_j\} \).

The recursive contract methodology of Marcet and Marimon (1998) ensures that the optimal solution to the optimization problem (14) satisfies \((c_{it}, i_{it}, \gamma_{it}) = \Psi(k_{it}, \mu_{it}, \theta_{it})\) for all \( t \) with the initial conditions \((k_{i0}, 0, \theta_{i0})\). In a particular case when symmetric treatment can be applied to the participants of the risk-sharing arrangement in (14) the reservation value on the right hand side of (13) is given by
\[ V^d(k_t, \theta_t) = \frac{1}{\lambda_j} W(k_t, 0, \theta_t). \]

Otherwise, this reservation value is simply the expected discounted sum of instantaneous utilities \( \sum_{t=0}^{\infty} \beta^t u(c_{it}) \) where \( \{c_{it}\} \) are allocations obtained from iterating on (17).

### 3.3 International borrowing under limited commitment and several types of capital.

Another example where the suggested algorithm will be useful is a framework with limited commitment with several types of capital. A version of this model is considered in Chapter 1 whereby the borrower which cannot fully commit to repay its debts accumulates capital in several productive sectors.

### 4 An Example of Implementation.

In this section we describe a practical computational strategy for implementing the algorithm. We will rely on the example in section 3.1 which considers international risk sharing problem of Backus, Kehoe, and Kydland (1992) augmented with the contract enforcement friction and habit formation preferences.

The intertemporal optimality condition to the planner’s problem (9) includes partial derivatives of the value function corresponding to the dynamic programming formulation of the agents outside option. The functional equation for the autarkic problem is given below:
\[ V(k, h, \theta) = \max_{(c, i) \in A(k, \theta)} \{ u(c, h) + \beta E[V(k', h', \theta') \mid (k, h, \theta)] \} \]

\[ h' = h + \lambda (c - h), \]

\[ k' = (1 - \delta) k + i \]

\[ A(k, \theta) = \{(c, i) \in \mathbb{R}_+^2 : c + i = f(k, \theta)\}, \]

The objective of the algorithm is to find the values of the partials in question \( V_h(\cdot) \) and \( V_k(\cdot) \) at a point \((\bar{k}, \bar{h}, \bar{\theta})\) which is likely to happen in equilibrium. Since the analytical expression for these derivatives is in general unavailable, we will have no choice but to rely on numerical differentiation. Another complication which arises here is that the closed form solution to the optimal value function is generally unavailable either. Hence, one needs to approximate value function at two points, e.g. \( V(\bar{k} + \varepsilon, \bar{h}, \bar{\theta}) \) and \( V(\bar{k} - \varepsilon, \bar{h}, \bar{\theta}) \) with arbitrary accuracy in order to be able to use finite differencing approach\(^7\).

The first step of the algorithm involves solving the model with a spectral method which can approximate the policy functions with arbitrary accuracy. In this example we will utilize a version of PEA of Marcet (1989) which allows us to formulate the solution in terms of approximated policy functions.

The Euler equation for the problem is given by:

\[ u_c(c, h) + \beta \lambda E_t \left[ \sum_{j=0}^{\infty} \beta^j (1 - \lambda)^j u_h(c_{t+j+1}, h_{t+j+1}) \right] = \beta E_t \left[ (u_c(c_{t+1}, h_{t+1}) + \beta \lambda \sum_{j=0}^{\infty} \beta^j (1 - \lambda)^j u_h(c_{t+j+2}, h_{t+j+2}) (f_k(k_{t+1}, \theta_{t+1}) + 1 - \delta) \right]. \]

To simplify the exposition we will consider the case of non-persistent habits which corresponds to \( \lambda = 1 \). In this particular case, the habit stock at \( t + 1 \) is simply the level of consumption at \( t \), and the Euler equation (18) reduces to:

\[ u_c(c_t, h_t) + \beta E_t \left[ u_h(c_{t+1}, h_{t+1}) \right] = \beta E_t \left[ (f_k(k_{t+1}, \theta_{t+1}) + 1 - \delta) \times (u_c(c_{t+1}, h_{t+1}) + \beta u_h(c_{t+2}, h_{t+2})) \right]. \]

To solve the model numerically we will assume the functional forms relatively standard in the

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\(^7\)The points of interest are \( V(\bar{k} + \varepsilon, \bar{h}, \bar{\theta}) \) and \( V(\bar{k} - \varepsilon, \bar{h}, \bar{\theta}) \) if we choose to rely on Newton’s forward formula, \( V(\bar{k}, \bar{h}, \bar{\theta}) \) if we use Newton’s backward formula, and \( V(\bar{k} + \varepsilon, \bar{h}, \bar{\theta}) \) and \( V(\bar{k} - \varepsilon, \bar{h}, \bar{\theta}) \) if we use Stirling’s formula.
growth literature. The instantaneous utility function is given by

\[ u(c_t, h_t) = \frac{(c_t - bh_t)^{1-\sigma}}{1 - \sigma}, \]

where \( b \in (0, 1) \) and \( \sigma > 0 \). One rationale for choosing additive functional form to introduce habits is to preserve the usual concavity properties of the utility function. The production function is Cobb-Douglas and is given by

\[ f(k_t, \theta_t) = \theta_t k_t^\alpha. \]

The stochastic productivity follows a first-order autoregressive process in logs

\[ \log \theta_t = \rho \log \theta_{t-1} + \varepsilon_t, \]

where \( \{\varepsilon_t\} \) are independent normally distributed random variables with zero mean and variance \( \sigma^2 \). In this example, we restrict attention to one particular set of the parameters which are summarized in Table 1.

[ Insert Table 1 about here. ]

With the chosen functional forms the Euler equation becomes:

\[ (c_t - bh_t)^{-\sigma} - \beta b E_t \left[ (c_{t+1} - bh_{t+1})^{-\sigma} \right] = \beta E_t \left[ \left( \alpha \theta_{t+1} k_{t+1}^\alpha + 1 - \delta \right) \times \left[ (c_{t+1} - bh_{t+1})^{-\sigma} - b \beta (c_{t+2} - bh_{t+2})^{-\sigma} \right] \right]. \]

The sequences of optimal allocations \( \{c_t, h_{t+1}, k_{t+1}\}_{t=0}^\infty \) must satisfy the following system of stochastic difference equations:

\[ (c_t - bh_t)^{-\sigma} = \beta E_t \left[ b (c_{t+1} - bh_{t+1})^{-\sigma} \times \left( 1 + \left( \alpha \theta_{t+1} k_{t+1}^\alpha - 1 \right) \left( \frac{1}{b} - \beta \left( \frac{c_{t+2} - bh_{t+2}}{c_{t+1} - bh_{t+1}} \right)^{-\sigma} \right) \right] \right], \]

\[ k_{t+1} = \theta_t k_t^\alpha + (1 - \delta) k_t - c_t, \]

\[ h_{t+1} = c_t. \]

For the expositional purpose, we will rely on the version of PEA which is easiest to implement. The computational procedure takes the following steps:
• Fix the initial conditions and draw a series of \( \{\theta_t\}_{t=1}^T \) that obeys the law of motion for the exogenous shocks with \( T \) sufficiently large.\(^8\)

• Substitute the conditional expectations in (20) by the flexible functional forms that depend on the state variables \( k_t, h_t, \theta_t \) and some coefficients to yield

\[
(c_t(\omega) - bh_t(\omega))^{-\sigma} = \beta \psi(\omega; k_t(\omega), h_t(\omega), \theta_t),
\]

where

\[
\psi(\omega; k_t(\omega), h_t(\omega), \theta_t) = \exp(P_n(\omega; \log k_t(\omega), \log h_t(\omega), \log \theta_t)),
\]

and \( P_n \) denotes polynomial of degree \( n \). Reliance on the exponent of the logarithmic polynomial expansion guarantees that the left hand side of (20) would be positive. Given \( c_t(\omega) \), the next period values for the capital and habit stocks follow directly from the corresponding laws of motion (21) and (22).

• Using the realizations of \( \{\theta_t\}_{t=0}^T \) repeat the previous step in order to obtain recursively a series of the endogenous variables \( \{c_t(\omega), k_{t+1}(\omega), h_{t+1}(\omega)\}_{t=0}^T \), for this particular parameterization \( \omega \).

• Run the following non-linear regression

\[
Y_t(\omega) = \exp(P_n(\xi; \log k_t(\omega), \log h_t(\omega), \log \theta_t)) + \eta_t,
\]

where the role of the dependent variable \( Y_t(\omega) \) is performed by the expression inside the conditional expectation in the RHS of (20).

• Letting \( S(\omega) \) be the result of the regression in the previous step, use an iterative procedure to find the fixed point of \( S \), and the set of coefficients \( \omega_f = S(\omega_f) \). This would provide the solution for the endogenous variables \( \{c_t(\omega_f), k_{t+1}(\omega_f), h_{t+1}(\omega_f)\}_{t=0}^T \) for this particular realization of the stochastic process \( \{\theta_t\}_{t=1}^T \) along with the approximated policy functions:

\[
c_t(k_t, h_t, \theta_t) = bh_t + [\beta \psi(\omega_f; k_t, h_t, \theta_t)]^{-\frac{1}{\sigma}},
\]

\[
k_{t+1}(k_t, h_t, \theta_t) = \theta_t k_t^\alpha + (1 - \delta)k_t - bh_t - [\beta \psi(\omega_f; k_t, h_t, \theta_t)]^{-\frac{1}{\sigma}},
\]

\[
h_{t+1}(k_t, h_t, \theta_t) = bh_t + [\beta \psi(\omega_f; k_t, h_t, \theta_t)]^{-\frac{1}{\sigma}}.
\]

The simulated series consistent with optimality and feasibility conditions are reported in

\(^8\)In order to ensure sufficient accuracy of the solution we chose \( T = 50,000 \) for all the numerical examples considered. The computational burden of this is still rather low since the model needs to be solved only once.
Figure 1. The initial values for capital, $k_0$, and habit stock, $h_0$ correspond to those of the deterministic steady state.

[Insert Figure 1 about here.]

Our objective is to find approximations of partials at a range of points. Supposing that the point of interest is $(\bar{k}, \bar{h}, \bar{\theta})$, the algorithm proceeds as follows:

- Simulate $N$ sequences of the realizations of the stochastic process $\{\boldsymbol{\theta}^n\}_{t=0}^T$ of size $T$ with a starting value $\theta_0^n = \bar{\theta}$, for all $n = 1, \ldots, N$. For each sequence $\{\theta^n\}_{t=0}^T$ simulate the series of the endogenous variables $\{k^n_t, h^n_t, c^n_t\}_{t=0}^T$ using approximated policy functions and the laws of motion for the state variables (23)-(25), and the corresponding initial values $k^n_0 = \bar{k}, h^n_0 = \bar{h}$. Using the simulated series calculate the discounted sums of the instantaneous utilities and average over $N$.

$$V(\bar{k}, \bar{h}, \bar{\theta}) \simeq \frac{1}{N} \sum_{n=1}^N \sum_{t=0}^T \beta^t \left( c^n_t - bh^n_t \right)^{1-\sigma}.$$  

- To obtain $V_k(\bar{k}, \bar{h}, \bar{\theta})$ get approximations of the optimal value function at $V(k + \epsilon, h, \theta)$ and $V(k - \epsilon, h, \theta)$, where $\epsilon$ is a small positive number. Calculate the approximated value of the partial derivative using Stirling’s finite difference formula:

$$\frac{\partial V(\bar{k}, \bar{h}, \bar{\theta})}{\partial k} \simeq \frac{V(\bar{k} + \epsilon, \bar{h}, \bar{\theta}) - V(\bar{k} - \epsilon, \bar{h}, \bar{\theta})}{2\epsilon}.$$  

The partial with respect to the habit stock is obtained in a similar way.

[Insert Figure 2 about here.]

Notice, that the length of the simulated series $T$ can be very moderate due to discounting of the future utilities. The optimal value of $\epsilon$ is both computer and problem specific. The approximations to the derivatives at a range of points are reported in Figure 2.

4.1 Comparing performance of the algorithm with the available alternatives

In this section we will consider issue of accuracy of our algorithm in the context of an example. First, we will compare performance of our algorithm with the approach of Marcet and Marimon (1992) when such comparison is feasible. Furthermore, we present several special
cases which allow us to isolate the contributions to the overall approximation error of the algorithm from different sources.

Once again, consider the risk-sharing problem with endogenous market incompleteness and habits discussed in section 3.1. Notice, that the optimality conditions for the autarkic problem (8) can be written in the sequence form as follows:

\[
\begin{align*}
\text{u}_c(c, h) + \beta \lambda E_t[V_h(k_{t+1}, h_{t+1}, \theta_{t+1})] &= \beta E_t[V_k(k_{t+1}, h_{t+1}, \theta_{t+1})], \\
V_k(k_t, h_t, \theta_t) &= \beta E_t[V_h(k_{t+1}, h_{t+1}, \theta_{t+1})] (f_k(k_t, \theta_t) + 1 - \delta), \\
V_h(k_t, h_t, \theta_t) &= \text{u}_h(c_t, h_t) + \beta (1 - \lambda) E_t[V_h(k_{t+1}, h_{t+1}, \theta_{t+1})].
\end{align*}
\]

The intertemporal condition (28) can be used to compare our algorithm with the Marcet and Marimon (1992) method. The latter requires solving the model numerically and expressing the derivatives of interest in terms of conditional expectations and functions of equilibrium path of the model. Starting from (28), using recursive substitution and the law of iterated expectations yields:

\[
V_h(k_t, h_t, \theta_t) = \text{u}_h(c_t, h_t) + \beta \sum_{j=1}^{\infty} \beta^j (1 - \lambda)^j \text{u}_h(c_{t+j}, h_{t+j}).
\]

Now, one can proceed by parameterizing the right hand side with flexible functional forms in the state variables \((k_t, h_t, \theta_t)\). An approximation of this derivative can be obtained by running one non-linear regression using the simulated series from the numerical solution of the model.

The approximations of the derivative obtained using our algorithm and the approach of Marcet and Marimon (1992) are reported in Figure 3. In the graphs, we plot the approximated values of \(V_h(k_t, h_t, \theta_t)\) for a range of one of the state variables while keeping the remaining ones fixed at their deterministic steady state values. The histograms plot the sample distributions of capital and habit stock from one long simulation of the model. Based on the graphs and histograms presented, a few observations can be made. First, the two algorithms produce indistinguishable results when the state variables take the values which often happen in equilibrium. Second, for the point which are unlikely to occur is equilibrium, the approximations differ significantly. To see this feature, consider the range of values of capital stock in excess of 6.5. The plots of the approximate derivatives reported in the upper panel of 3 do not coincide. Moreover, the upper tail of the histogram suggests that such values of \(k_t\) are not unlikely to happen in equilibrium. Notice, that while considering at a relatively high value of \(k_t\) we kept the remaining arguments of \(V_h(k_t, h_t, \theta_t)\) at their deterministic steady state values. However,
the points where capital is very high while consumption (and hence habit stock) are at the steady state level are rather unusual. This can be also noticed from the graph in Figure 1 which plots the evolution of the endogenous variables and exogenous shocks for 200 periods corresponding to the stationary distribution. This is an expected result, since we relied on a version of PEA which delivers good approximation to the policy function in the region of the state space which is frequently visited by the model in equilibrium. This by no means limits the applicability of the algorithm. If one happens to be interested in a different subset of the state space one can simply modify the oversampling scheme in Step I of the algorithm. We have followed this strategy in Dmitriev (2008) where the object of interest was the transition path towards the stationary distribution.

The framework we have chosen to serve as worked out example embeds several well known special cases. For instance, for $\lambda = 0$, it reduces to the stochastic growth model of Brock and Mirman (1972). In this case, the analytical form of the one-period return function $r$ which maps the graph $A$ of the feasibility correspondence $\Gamma$ into the real numbers is known. Indeed, the correspondence describing the feasibility constraints is given by

$$\Gamma(k_t, \theta_t) = [f(1 - \delta)k_t, f(k_t, \theta_t) + (1 - \delta)k_t],$$

and the instantaneous return function becomes

$$r : A \rightarrow \mathbb{R} \text{ given by } r(k_t, k_{t+1}, \theta_t) = u(f(k_t, \theta_t) + (1 - \delta)k_t - k_{t+1}),$$

where $A = \{(k_t, k_{t+1}, \theta_t) \in \mathbb{R}^3 : k_{t+1} \in \Gamma(k_t, \theta_t)\}$. Hence, by virtue of the Benveniste-Sheinkman theorem the derivative of interest can be expressed as

$$V_k(k_t, \theta_t) = u'(f(k_t, \theta_t) + (1 - \delta)k_t - g(k_t, \theta_t)) [f_k(k_t, \theta_t) + 1 - \delta],$$

where $g$ is the optimal policy function for capital stock. This special case allows us to compare the simulation from our algorithm with the example where the only source of approximation errors is the approximation of the policy function $g$. This will allow us to isolate the contribution of the approximation errors in evaluation of the integrals and numerical differentiation to the overall approximation error of the algorithm. As can be seen from Figure 4 the approximations delivered by our algorithm are very close to the approximation which rely on the Benveniste-Sheinkman theorem. Once again, in the region of the state space which is often visited by the model in equilibrium the two approximations are virtually identical. This allows us to tentatively suggest that the main contribution to the approximation error of the algorithm comes from the approximation of the policy functions.

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Our final special case allows us to compare the approximation of the derivative with its known exact solution. It is well known that for the functional forms \( f(k_t, \theta_t) = \theta_t k_t^\alpha \), \( u(c_t) = \log c_t \), and \( \delta = 1 \), the optimal policy function is defined by the simple law of motion \( k_{t+1} = \alpha \beta \theta_t k_t^\alpha \). Moreover, the derivative of the value function has the following analytical solution:

\[
V_k(k_t, \theta_t) = \frac{\alpha \theta_t k_t^{\alpha-1}}{(1 - \alpha \beta) \theta_t k_t^\alpha} = \frac{\alpha}{(1 - \alpha \beta)} \frac{1}{k_t}.
\]

Notice that by replacing the approximated policy function with the known closed form solution, we can isolate the effect of the errors stemming from Monte Carlo integration and numerical differentiation on the accuracy of the approximation. In Figure 5 we compare the approximated derivatives obtained using exact policy function for \( k_t \) with the graph of the analytical derivative. The reported graphs are visually indistinguishable for the whole range considered, i.e. six standard deviations of \( k_t \) around its deterministic steady state value. The approximation errors stemming from Monte Carlo integration and numerical differentiation are of order of \( 10^{-9} \) of the value of the derivative. This suggests that obtaining accurate approximation of the policy functions in the region of state space of interest is crucial for the accuracy of the whole algorithm.

5 Concluding Remarks

The purpose of the note was to present an algorithm for computing the partial derivatives of the optimal value function by simulation. The procedure proposed is conceptually straightforward, computationally inexpensive, and simple to implement. Yet, it is flexible enough to handle dynamic model with a large number of state variables even when derivatives of interest cannot be expressed in terms of conditional expectations and functions of equilibrium path of the model. For our benchmark examples the algorithm has a performance comparable with approximation method introduced by Marcet and Marimon (1992). Furthermore, the algorithm can be applied to some problems which fail to fall into standard recursive framework and for which Bellman equation is not well defined.
References


6 Appendix: Derivations

6.1 Optimality conditions for example in section (3.1).

Since the constraint (7) involves expected values of the future decision variables, this problem is not a special case of the standard dynamic programming problems, and the Bellman equation will not be satisfied. However, as shown by Marcet and Marimon (1998) it falls into a general class of problems, which can be cast into an alternative recursive framework. The recursive saddle point problem associated with dynamic problem above will be given by

\[
\max_{\{c_{it},i_{it}\}} \min_{\{\mu_{it}\}} \sum_{t=0}^{\infty} \beta^t \left\{ \left( \lambda_{i_{t}} + M_{it} \right) u(c_{it}, h_{it}) + \mu_{it} (u(c_{it}, h_{it}) - V^a_{it}(k_{it}, h_{it}, \theta_{it})) \right\}
\]

subject to (4)-(7) and

\[
M_{it+1} = M_{it} + \mu_{it}, \quad M_{i0} = 0,
\]

\[
\mu_{it} \geq 0.
\]

Indeed, the corresponding Lagrangian is

\[
\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \sum_{i=1}^{I} \lambda_{i} u(c_{it}, h_{it}) + \sum_{i=1}^{I} \mu_{it} \left( E_t \sum_{j=0}^{\infty} \beta^j u(c_{it+j}, h_{it+j}) - V^a_{it}(k_{it}, h_{it}, \theta_{it}) \right) \right\}
\]

subject to (4)-(6), given \( \mu_{it} \geq 0 \), where \( \beta^{-t} \mu_{it} \) is the Lagrange multiplier of (7) at \( t \). The law of iterated expectations allows to imbed the conditional expectations \( E_t \) into \( E_0 \). Furthermore, reordering the terms and introducing the law of motion for \( M_{it} \) yields the above result.

As shown by Marcet and Marimon (1998), under certain assumptions the solution to the recursive saddle point problem obeys a saddle point functional equation. Within our framework their result implies that there exists a unique value function,

\[
W(k, h, M, \theta) = \min_{\{\mu_{it}\}} \max_{\{c_{it},i_{it}\}} \left\{ \sum_{i=1}^{I} \left[ (\lambda_{i} + M_{i}) u(c_{i}, h_{i}) + \mu_{i} (u(c_{i}, h_{i}) - V^a_{i}(k_{i}, h_{i}, \theta_{i})) \right] \right. \\
+ \left. \beta E [W(k', h', M', \theta') | \theta] \right\}
\]

subject to

\[
\sum_{i=1}^{I} c_{i} + \sum_{i=1}^{I} i_{i} = \sum_{i=1}^{I} f(k_{i}, \theta_{i}), \quad (30)
\]

\[
k'_{i} = (1 - \delta)k_{i} + i_{i}, \quad (31)
\]
\[ h_i' = h_i + \lambda (c_i - h_i), \]  
\[ M_i' = M_i + \mu_i, \]  
\[ c_i, i, \mu_i \geq 0, \]

for all \((k, h, M, \theta)\) and such that \(W(k_0, h_0, M_0, \theta_0)\) is the value of the optimization problem in question. The policy correspondence associated with the above saddle point functional equation is given by

\[
\psi(k, h, M, \theta) \in \arg \min_{(\mu_i')} \max_{(c_i, i')} \left\{ \left\{ \sum_{i=1}^{I} [(\lambda_i + M_i) u(c_i, h_i) + \mu_i (u(c_i, h_i) - V_i^a (k_i, h_i, \theta_i))] \right. \right.
\]
\[
\left. + \beta E [W(k', h', M', \theta') | \theta'] \right\} \}
\]

subject to (30) - (34).

The key results demonstrated by Marcet and Marimon (1998) ensures that the optimal solution to the optimization problem we consider satisfies \((c_t, i_t, \mu_t, \theta_t) = \psi(k_t, h_t, M_t, \theta_t)\) for all \(t\) with the initial conditions \((k_0, h_0, 0, 0)\). That is there exist a time invariant policy correspondence \(\psi\) such that only the values of a small number of past variables \((k_t, h_t, M_t, \theta_t)\) matter. Hence, the problem is now in a recursive framework the solution to which can now be obtained from studying the saddle point functional equation.

Denoting by \(m_{it}, n_{it}\) and \(\gamma_t\) the Lagrange multipliers of the constraints (5), (6) and (4), the first order conditions for this problem become:

\[
(\lambda_i + M_{it} + \mu_{it}) \frac{\partial u(c_{it}, h_{it})}{\partial c_{it}} + \lambda n_{it} - \gamma_{it} = 0
\]

\[
m_{it} - \gamma_{it} = 0
\]

\[
\beta E_t \left[ -\mu_{it+1} \frac{\partial V_i^a (k_{it+1}, h_{it+1}, \theta_{it+1})}{\partial k_{it+1}} + m_{it+1}(1 - \delta) + \gamma_{it+1} \frac{\partial f (k_{it+1}, h_{it+1}, \theta_{it+1})}{\partial k_{it+1}} \right] = m_{it}
\]

\[
\beta E_t \left[ (\lambda_i + M_{it+1} + \mu_{it+1}) \frac{\partial u(c_{it+j}, h_{it+j})}{\partial h_{it+1}} - \mu_{it+1} \frac{\partial V_i^a (k_{it+1}, h_{it+1}, \theta_{it+1})}{\partial h_{it+1}} + (1 - \lambda) n_{it+1} \right] = n_{it}
\]

\[
\mu_{it} \left[ E_t \sum_{j=0}^{\infty} \beta^j u(c_{it+j}, h_{it+j}) - V_i^a (k_{it}, h_{it}, \theta_{it}) \right] = 0
\]

\[
E_t \sum_{j=0}^{\infty} \beta^j u(c_{it+j}, h_{it+j}) \geq V_i^a (k_{it}, h_{it}, \theta_{it})
\]
in addition to the aggregate resource constraint (4), the laws of motion (5)-(6) for state variables, the laws of motion (29) for the co-state variables \(M_{it}\), and non-negativity of the
Lagrange multipliers $\mu_{it} \geq 0$.

$$
(\lambda_i + M_{it} + \mu_{it}) \frac{\partial u(c_{it}, h_{it})}{\partial c_{it}} + \lambda n_{it} = (\lambda_j + M_{jt} + \mu_{jt}) \frac{\partial u(c_{jt}, h_{jt})}{\partial c_{jt}} + \lambda n_{jt}, \text{ for } i, j = 1, ..., I,
$$

$$
m_{it} = \beta E_t \left[ m_{it+1} \left( \frac{\partial f(k_{it+1}, \theta_{it+1})}{\partial k_{it+1}} + 1 - \delta \right) - \mu_{it+1} \frac{\partial V^a_{i}}{\partial k_{it+1}} \right],
$$

$$
n_{it} = \beta (1 - \lambda) E_t [n_{it+1}] + \beta E_t \left[ (\lambda_i + M_{it+1} + \mu_{it+1}) \frac{\partial u(c_{it+1}, h_{it+1})}{\partial h_{it+1}} - \mu_{it+1} \frac{\partial V^a_{i}}{\partial h_{it+1}} \right].
$$

Using recursive substitution and the law of iterated projections yields

$$
n_{it} = \beta E_t \sum_{j=0}^{\infty} \beta^j (1 - \lambda)^j \left[ \frac{\xi_{it+1+j}}{\xi_{it}} \frac{\partial u(c_{it+1+j}, h_{it+1+j})}{\partial h_{it+1+j}} - \mu_{it+1+j} \frac{\partial V^a_{i}}{\partial h_{it+1+j}} \right],
$$

where the time varying planner weights are given by

$$
\xi_{it} = \lambda_i + M_{it} + \mu_{it}.
$$

Re-organizing the terms yields the following optimality conditions:

$$
\frac{u_c(i, t) + \lambda \beta E_t \sum_{j=0}^{\infty} \beta^j (1 - \lambda)^j \left[ \frac{\xi_{it+1+j}}{\xi_{it}} \frac{\partial u(c_{it+1+j}, h_{it+1+j})}{\partial h_{it+1+j}} - \mu_{it+1+j} \frac{\partial V^a_{i}}{\partial h_{it+1+j}} \right]}{u_c(s, t) + \lambda \beta E_t \sum_{j=0}^{\infty} \beta^j (1 - \lambda)^j \left[ \frac{\xi_{st+1+j}}{\xi_{st}} \frac{\partial u(c_{st+1+j}, h_{st+1+j})}{\partial h_{st+1+j}} - \mu_{st+1+j} \frac{\partial V^a_{i}}{\partial h_{st+1+j}} \right]} = \frac{\xi_{st}}{\xi_{it}},
$$

for $i, s = 1, ..., I$.

Optimal allocation must also satisfy the following intertemporal condition:

$$
\frac{u_c(i, t) + \lambda \beta E_t \sum_{j=0}^{\infty} \beta^j (1 - \lambda)^j \left[ \frac{\xi_{it+1+j}}{\xi_{it}} \frac{\partial u(c_{it+1+j}, h_{it+1+j})}{\partial h_{it+1+j}} - \mu_{it+1+j} \frac{\partial V^a_{i}}{\partial h_{it+1+j}} \right]}{u_c(s, t) + \lambda \beta E_t \sum_{j=0}^{\infty} \beta^j (1 - \lambda)^j \left[ \frac{\xi_{st+1+j}}{\xi_{st}} \frac{\partial u(c_{st+1+j}, h_{st+1+j})}{\partial h_{st+1+j}} - \mu_{st+1+j} \frac{\partial V^a_{i}}{\partial h_{st+1+j}} \right]} = \frac{\xi_{st}}{\xi_{it}},
$$

$$
\beta E_t \left[ \left( \frac{\xi_{it+1+j}}{\xi_{it}} u_c(i, t+1) + \lambda \beta \sum_{j=0}^{\infty} \beta^j (1 - \lambda)^j \left[ \frac{\xi_{it+1+j}}{\xi_{it}} \frac{\partial u(c_{it+1+j}, h_{it+1+j})}{\partial h_{it+1+j}} - \mu_{it+1+j} \frac{\partial V^a_{i}}{\partial h_{it+1+j}} \right] \right) (f_k(k_{it+1}, \theta_{it+1}) + 1 - \delta) \right] = \frac{\mu_{it+1+j}}{\xi_{it}} \frac{\partial V^a_{i}}{\partial k_{it+1+j}} (i, t+1).
$$

In these first order conditions we have used the abbreviations $u_c(i, t)$ for $\frac{\partial u(c_{it}, h_{it})}{\partial c_{it}}$, and we have used similar abbreviations for other terms.
6.2 Optimality conditions for stochastic growth model with habits.

Consider the following dynamic optimization problem corresponding to the stochastic growth model with additive habit formation preferences:

\[
\max_{\{c_t, i_t\}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, h_t)
\]

subject to

\[
c_t + i_t = f(k_t, \theta_t), \quad (35)
\]

\[
k_{t+1} = (1 - \delta) k_t + i_t, \quad (36)
\]

\[
h_{t+1} = h_t + \lambda (c_t - h_t), \quad (37)
\]

with the initial conditions \(k_0, h_0\) given.

Using the arguments of standard dynamic programming (see Stokey, Lucas, and Prescott (1989)) one can show the existence of the time invariant policy functions \(c(k, h, \theta), i(k, h, \theta)\) and a value function \(V(k, h, \theta)\). The functional equation for the problem is given by

\[
V(k, h, \theta) = \max_{(c, i) \in A(k)} \{u(c, h) + \beta E[V(k', h', \theta') \mid (k, h, \theta)]\}
\]

\[
h' = h + \lambda (c - h),
\]

\[
k' = f(k, \theta) + (1 - \delta) k + i,
\]

\[
A(k, \theta) = \{ (c, i) \in \mathbb{R}^2_+ : c + i = f(k, \theta) \}.
\]

The first order condition of the maximization problem in the right hand side of the functional equation is given by

\[
uc(c, h) = \beta E[V_k(k', h', \theta') - \lambda V_h(k', h', \theta') \mid (k, h, \theta)].
\]  

(38)

Let \(g(k, h, \theta)\) be the optimal policy function for investment. Then, the following identity must hold:

\[
V(k, h, \theta) = u(f(k, \theta) - g(k, h, \theta), h) + \beta E[V((1 - \delta)k + g(k, h, \theta), (1 - \lambda) h + \lambda (f(k, \theta) - g(k, h, \theta)), \theta') \mid (k, h, \theta)].
\]
Differentiating both sides of the equality above with respect to $k$ yields

$$V_k(k, h, \theta) = u_c(c, h) [f_k(k, \theta) - g_k(k, h, \theta)]$$

$$+ \beta E [V_k(k', h', \theta') (1 - \delta) + g_k(k, h, \theta)]$$

$$+ \lambda V_h(k', h', \theta') [f_k(k, \theta) - g_k(k, h, \theta)] | (k, h, \theta),$$

which using (38) reduces to

$$V_k(k, h, \theta) = \beta E [V_k(k', h', \theta') | (k, h, \theta)].$$

Differentiating both sides of the identity above with respect to $h$ yields

$$V_h(k, h, \theta) = u_h(c, h) - u_c(c, h)g_h(k, h, \theta) + \beta E [V_h(k', h', \theta') g_k(k, h, \theta)$$

$$+ V_h(k', h', \theta') [(1 - \lambda) - \lambda g_h(k, h, \theta)] | (k, h, \theta)],$$

which using (38) reduces to

$$V_h(k, h, \theta) = u_h(c, h) + \beta (1 - \lambda) E [V_h(k', h', \theta') | (k, h, \theta)].$$

The optimality conditions can be written in the sequence form in the following way:

$$u_c(c, h) + \beta \lambda E_t [V_h(k_{t+1}, h_{t+1}, \theta_{t+1})] = \beta E_t [V_k(k_{t+1}, h_{t+1}, \theta_{t+1})],$$

(39)

$$V_k(k_t, h_t, \theta_t) = \beta E_t [V_k(k_{t+1}, h_{t+1}, \theta_{t+1})] (f_k(k_t, \theta_t) + 1 - \delta),$$

(40)

$$V_h(k_t, h_t, \theta_t) = u_h(c_t, h_t) + \beta (1 - \lambda) E_t [V_h(k_{t+1}, h_{t+1}, \theta_{t+1})].$$

(41)

Combining (39) and (40) yields the following expression for derivative of the value function w.r.t. the capital stock:

$$V_k(k_t, h_t, \theta_t) = (u_c(c_t, h_t) + \beta \lambda E_t [V_h(k_{t+1}, h_{t+1}, \theta_{t+1})]) (f_k(k_t, \theta_t) + 1 - \delta)$$

Shifting the expression above one period ahead and substituting in into (40) gives

$$u_c(c, h) + \beta \lambda E_t [V_h(k_{t+1}, h_{t+1}, \theta_{t+1})] = \beta E_t [(f_k(k_{t+1}, \theta_{t+1}) + 1 - \delta)$$

$$\times (u_c(c_{t+1}, h_{t+1}) + \beta \lambda V_h(k_{t+2}, h_{t+2}, \theta_{t+2}))].$$

Using recursive substitution and the law of iterated expectations (41) provides an expression
for the derivative of the value function w.r.t. the habit stock:

\[ V_h(k_t, h_t, \theta_t) = u_h(c_t, h_t) + E_t \left[ \sum_{j=1}^{\infty} \beta^j (1 - \lambda)^j u_h(c_{t+j}, h_{t+j}) \right]. \]

Finally, the Euler equation for the problem becomes:

\[ u_c(c, h) + \beta \lambda E_t \left[ \sum_{j=0}^{\infty} \beta^j (1 - \lambda)^j u_h(c_{t+j+1}, h_{t+j+1}) \right] = \beta E_t \left[ f_k(k_{t+1}, \theta_{t+1}) + 1 - \delta \right] \]

\[ \times \left( u_c(c_{t+1}, h_{t+1}) + \beta \lambda \sum_{j=0}^{\infty} \beta^j (1 - \lambda)^j u_h(c_{t+j+2}, h_{t+j+2}) \right). \]
Figure 1: Stochastic growth model with habits: a numerical solution with PEA
Figure 2: Growth model with habits: approximation of the value function and its partial derivatives.
Figure 3: Alternative methods for the approximation of the derivatives

Marcet & Marimon (1992)
Figure 4: Stochastic growth model with separable preferences: simulations
Table 1: Parameterization of the model

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<th>Parameter</th>
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